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Networks Through SPECTRAL LENSES: the Laplacian Renormalization Group

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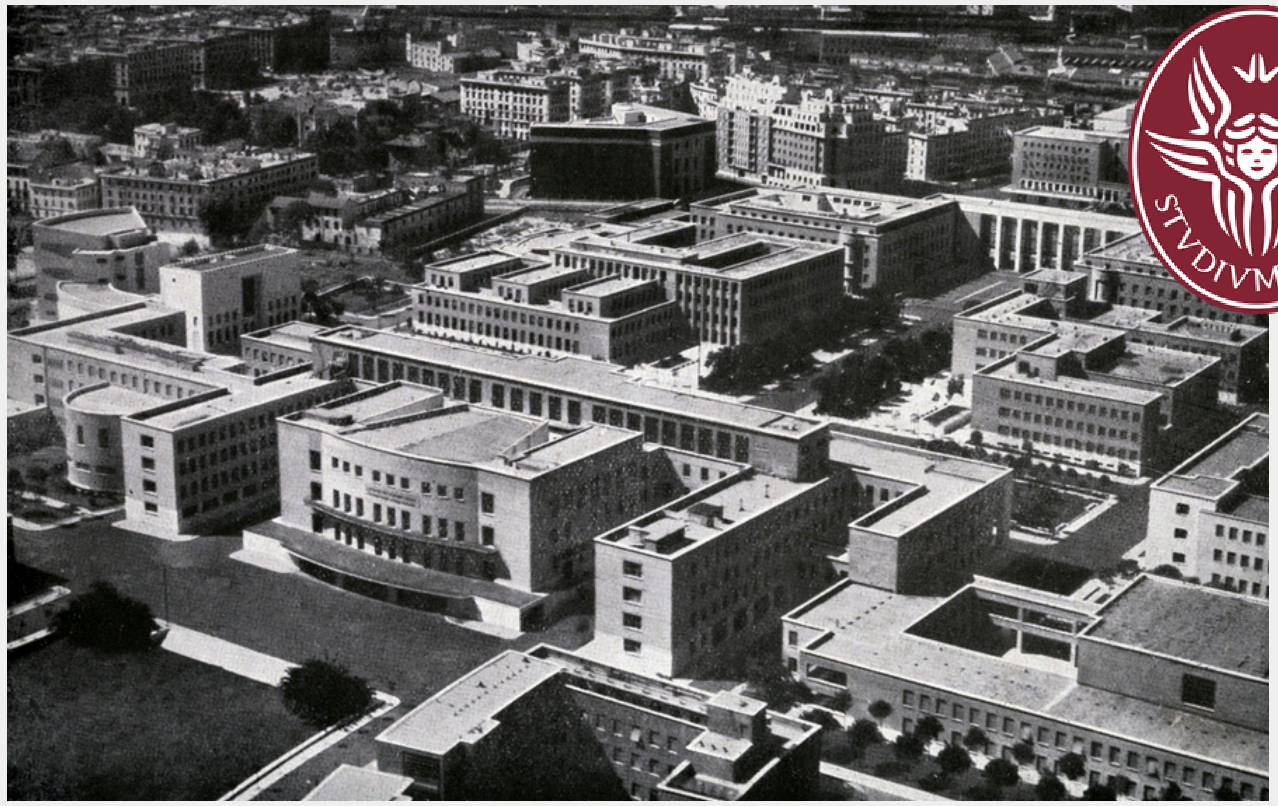
CENTRO RICERCHE ENRICO FERMI



SAPIENZA
UNIVERSITÀ DI ROMA

2015-2021

- Opinion dynamics with Recommendation Algorithms

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UNIVERSITÀ DEGLI STUDI DI ROMA

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2021-2024

- “Non-ergodicity” (memory) in Complex Systems

UNIVERSITÀ DEGLI STUDI DI PALERMO

2024-2025

- Higer-order features of human brain networks

- Complexity Physics in climate models

2025-2027

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The journey, Not the destination matters…
T.S. Eliot

On the Complex Systems' structure

What a network is?

Spectral Theory

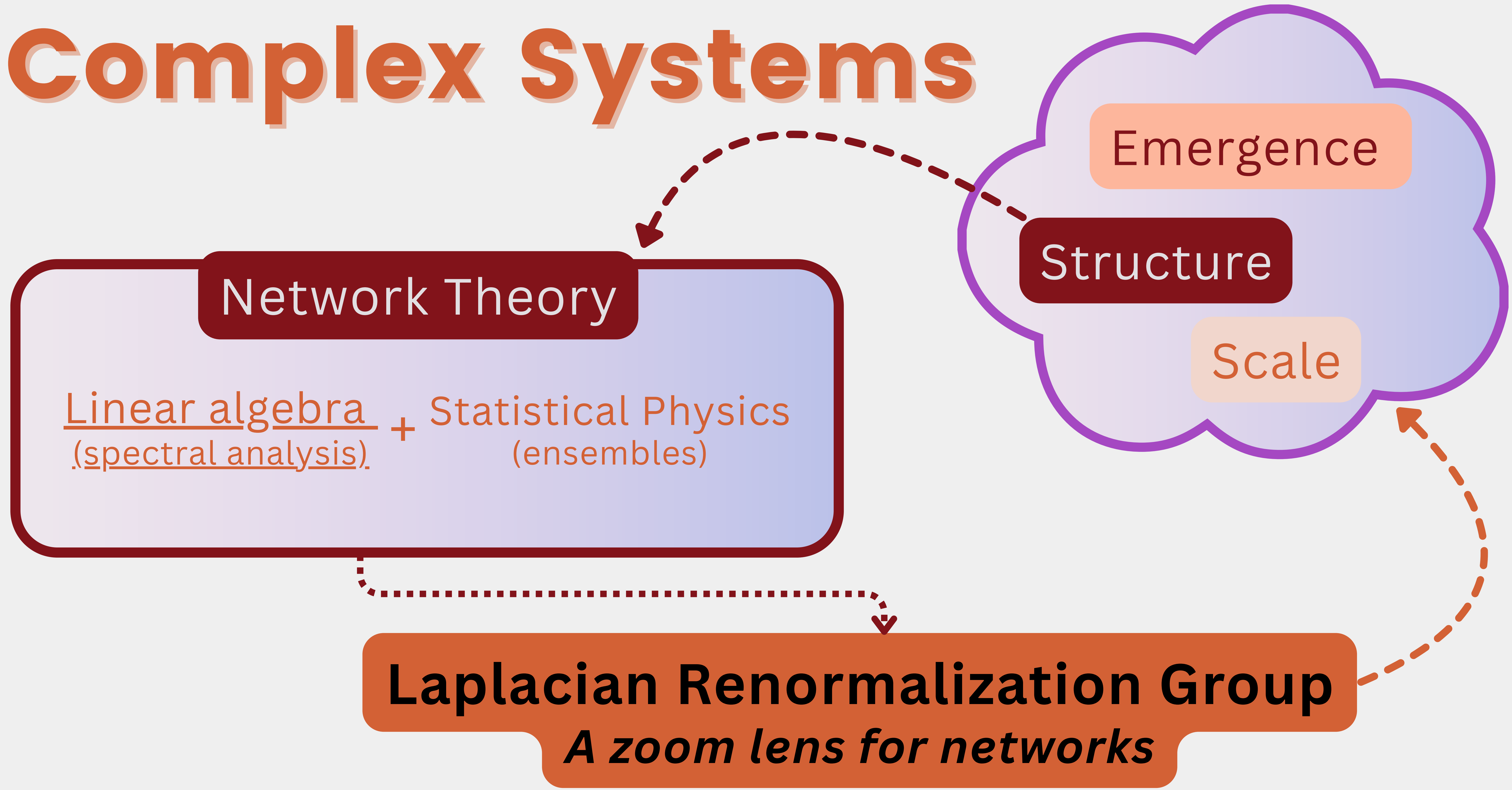
The diffusion process

The ensemble of diffusion

A zoom lens for networks

(Multiscale) Community Detection

Applications to brain networks



Complex Systems
Emergence
Structure
Scale
Network Theory
Linear algebra
(spectral analysis)
+
Statistical Physics
(ensembles)
Laplacian Renormalization Group
A zoom lens for networks

What is a Complex System?

No unique one-size-fits-all definition, but...

Emergence

"="

Interacting units

Interaction rules

\+

"Large" ***Phase space***

The problem with scale

- Emergence give rise to variety.
- Each scale add new behaviors and rules
- Fundamental problem: link microscopic laws with emergent macroscopic behavior

- R*enormalization schemes* (*coarse-graining + rescaling)* in Statistical Physics
  a. explanation of emergent behaviors from microscopic models
  b. multiscale approaches

More Is Different

Broken symmetry and the nature of the hierarchical structure of science.

P. W. Anderson

The reductionist hypothesis may still be a topic for controversy among philosophers, but among the great majority of active scientists I think it is accepted without question. The workings of our minds and bodies, and of all the animate or inanimate matter of which we have any detailed knowledge, are assumed to be controlled by the same set of fundamental laws, which except under certain extreme conditions we feel we know pretty well.

It seems inevitable to go on uncritically to what appears at first sight to be an obvious corollary of reductionism: that if everything obeys the same fundamental laws, then the only scientists who are studying anything really fundamental are those who are working on those laws. In practice, that amounts to some astrophysicists, some elementary particle physicists, some logicians and other mathematicians, and few planation of phenomena in terms of known fundamental laws. As always, distinctions of this kind are not unambiguous, but they are clear in most cases. Solid state physics, plasma physics, and perhaps also biology are extensive. High energy physics and a good part of nuclear physics are intensive. There is always much less intensive research going on than extensive. Once new fundamental laws are discovered, a large and ever increasing activity begins in order to apply the discoveries to hitherto unexplained phenomena. Thus, there are two dimensions to basic research. The frontier of science extends all along a long line from the newest and most modern intensive research, over the extensive research recently spawned by the intensive research of yesterday, to the broad and well developed web of extensive research activities based on intensive research of past decades.

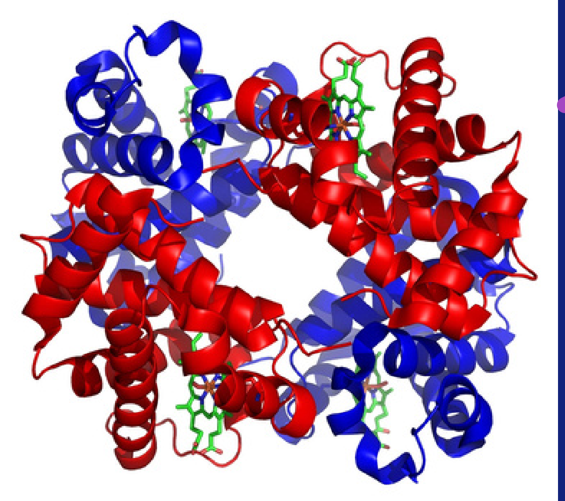
The effectiveness of this message may be indicated by the fact that I heard it quoted recently by a leader in the field of materials science, who urged the

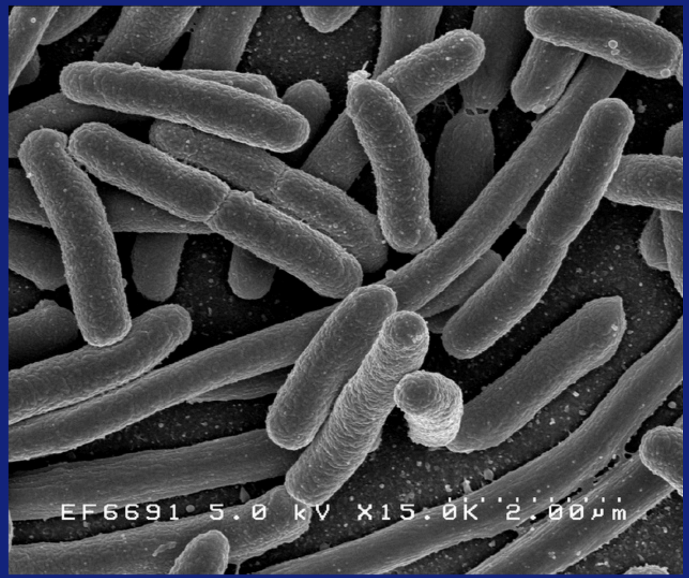
The constructionist hypothesis breaks down when confronted with the twin difficulties of scale and complexity. The behavior of large and complex aggregates of elementary particles, it turns out, is not to be understood in terms of a simple extrapolation of the properties of a few particles. Instead, at each level of complexity entirely new properties appear, and the understanding of the new behaviors requires research which I think is as fundamental in its nature as any other. That is, it seems to me that one may array the sciences roughly linearly in a hierarchy, according to the idea: The elementary entities of science X obey the laws of science Y.

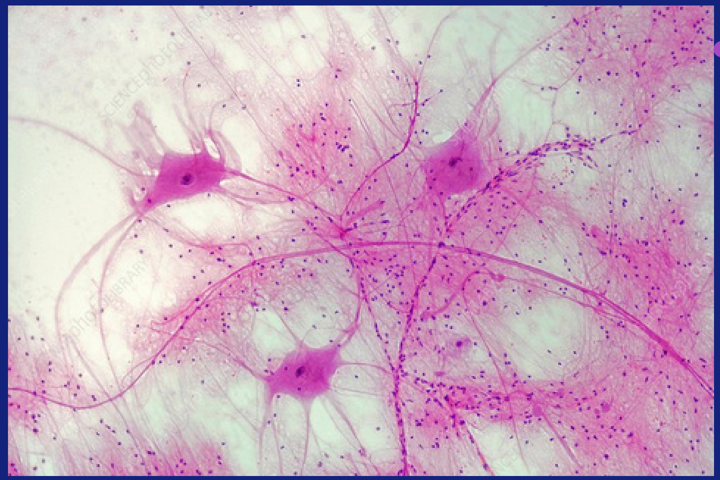
X	Y
solid state or many-body physics	elementary particle physics
chemistry	many-body physics
molecular biology	chemistry
cell biology	molecular biology
•	•
•	•
•	•
psychology	physiology
social sciences	psychology

But this hierarchy does not imply that science X is "just applied Y." At each stage entirely new laws, concepts, and generalizations are necessary, requiring inspiration and creativity to just as great a degree as in the previous one. Psychology is not applied biology, nor is biology applied chemistry.

Structure







Contituents

Fundamental Physics potentials

$$U(\{r\}) = \sum_{\langle ij\rangle \in \text{bonds}} \frac{k_{ij}}{2}\left(r_{ij} - r_{ij}^0\right)^2 + \sum_{i<j}^{\text{nonbonded}} 4\varepsilon_{ij}\left[\left(\frac{\sigma_{ij}}{r_{ij}}\right)^{12} - \left(\frac{\sigma_{ij}}{r_{ij}}\right)^{6}\right] + \sum_{i<j} \frac{q_i q_j}{4\pi\varepsilon_0 r_{ij}}.$$

Population dynamics

$$\dot{N}(t) = r\,N(t)\left(1 - \frac{N(t)}{K}\right).$$

Neuronal Excitation/Inhibition

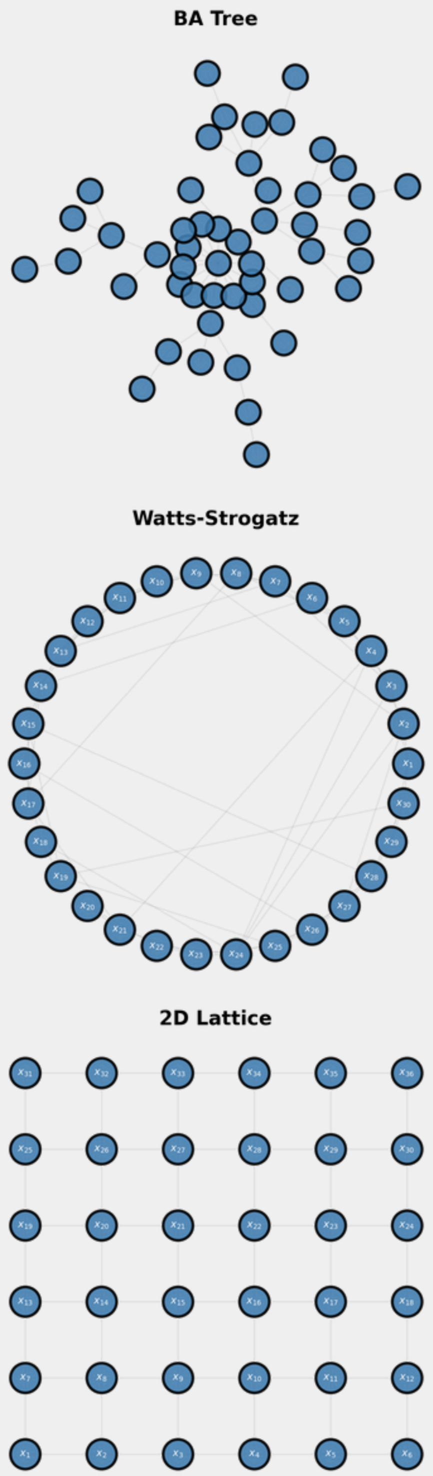
$$\tau_E\,\dot{E}(t) = -E(t) + \phi\big(w_{EE}E(t) - w_{EI}I(t) + P(t)\big),$$
$$\tau_I\,\dot{I}(t) = -I(t) + \phi\big(w_{IE}E(t) - w_{II}I(t) + Q(t)\big),$$
$$\phi(x) = \frac{1}{1 + e^{-a(x-\theta)}}.$$

Opinion dynamics

$$w_i(s) = \frac{1}{k_i}\sum_{j \in \mathcal{N}_i} \mathbf{1}[s_j \neq s_i],$$

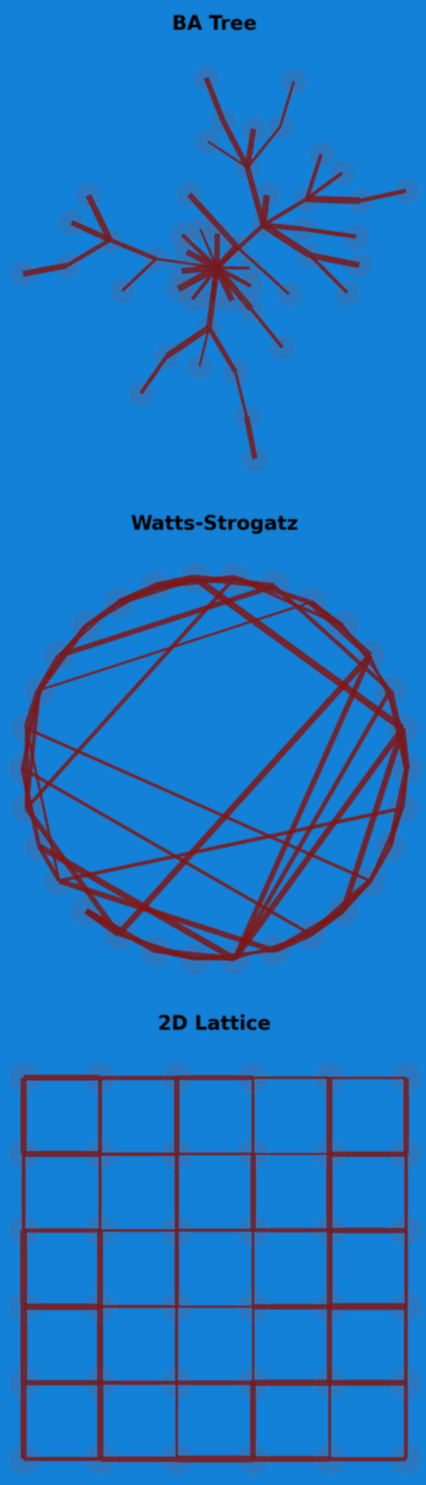
Rules

x_i : state variable



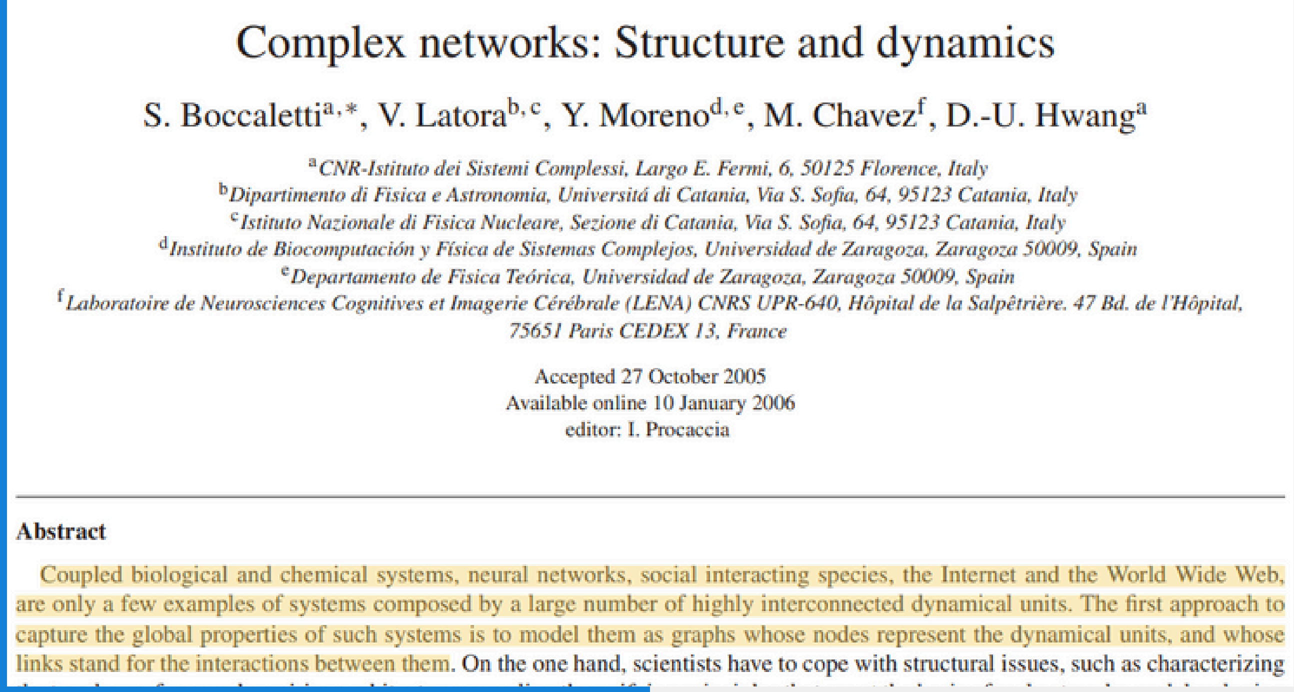
Nodes

A_{ij} : connection



Edges

Structure


Complex networks: Structure and dynamics

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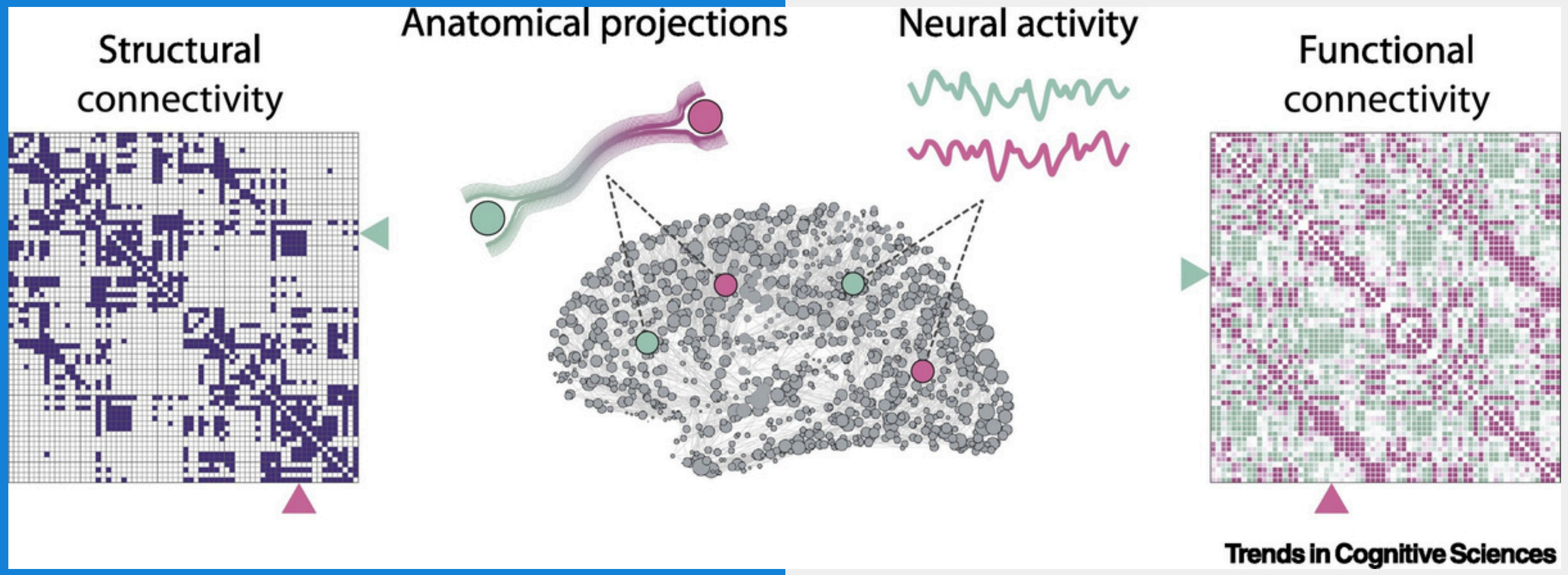
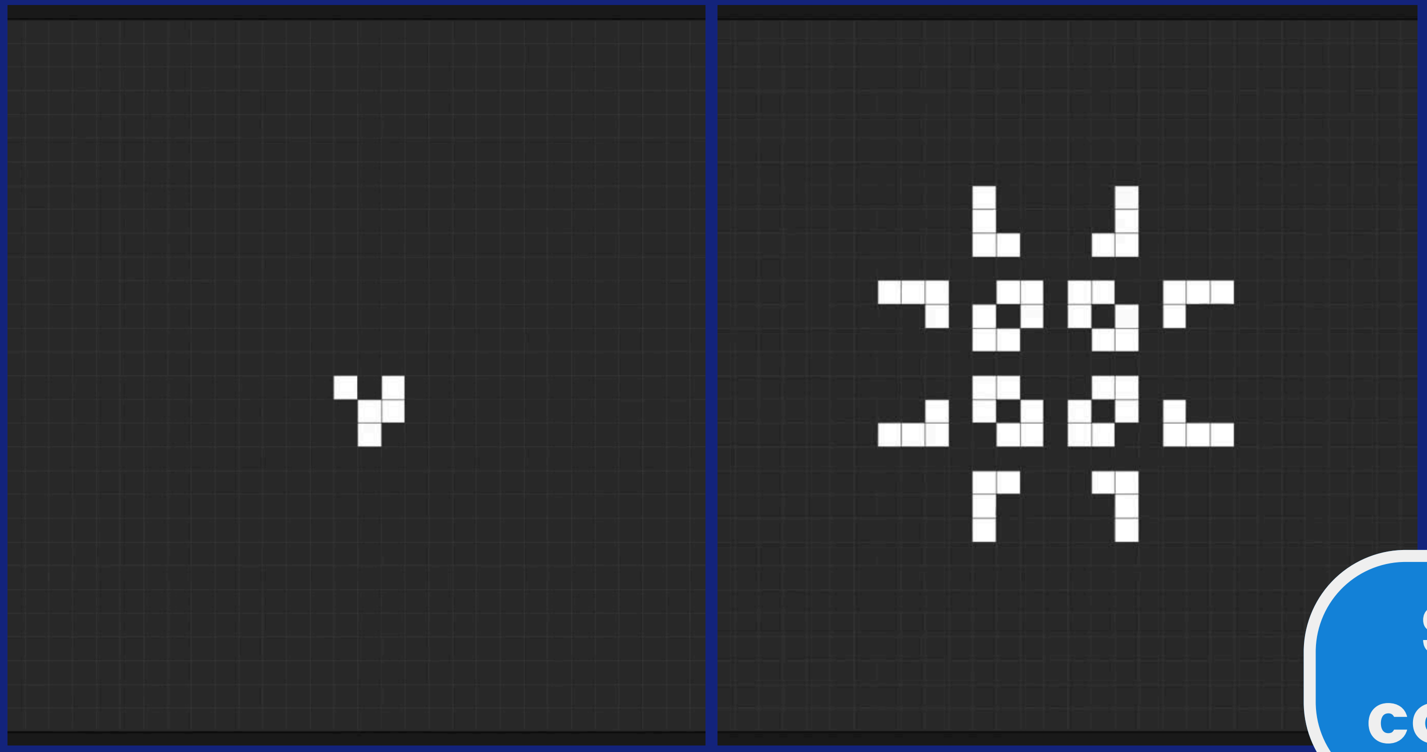
Abstract

Coupled biological and chemical systems, neural networks, social interacting species, the Internet and the World Wide Web, are only a few examples of systems composed by a large number of highly interconnected dynamical units. The first approach to capture the global properties of such systems is to model them as graphs whose nodes represent the dynamical units, and whose links stand for the interactions between them. On the one hand, scientists have to cope with structural issues, such as characterizing

Dynamics



Conway's game of life



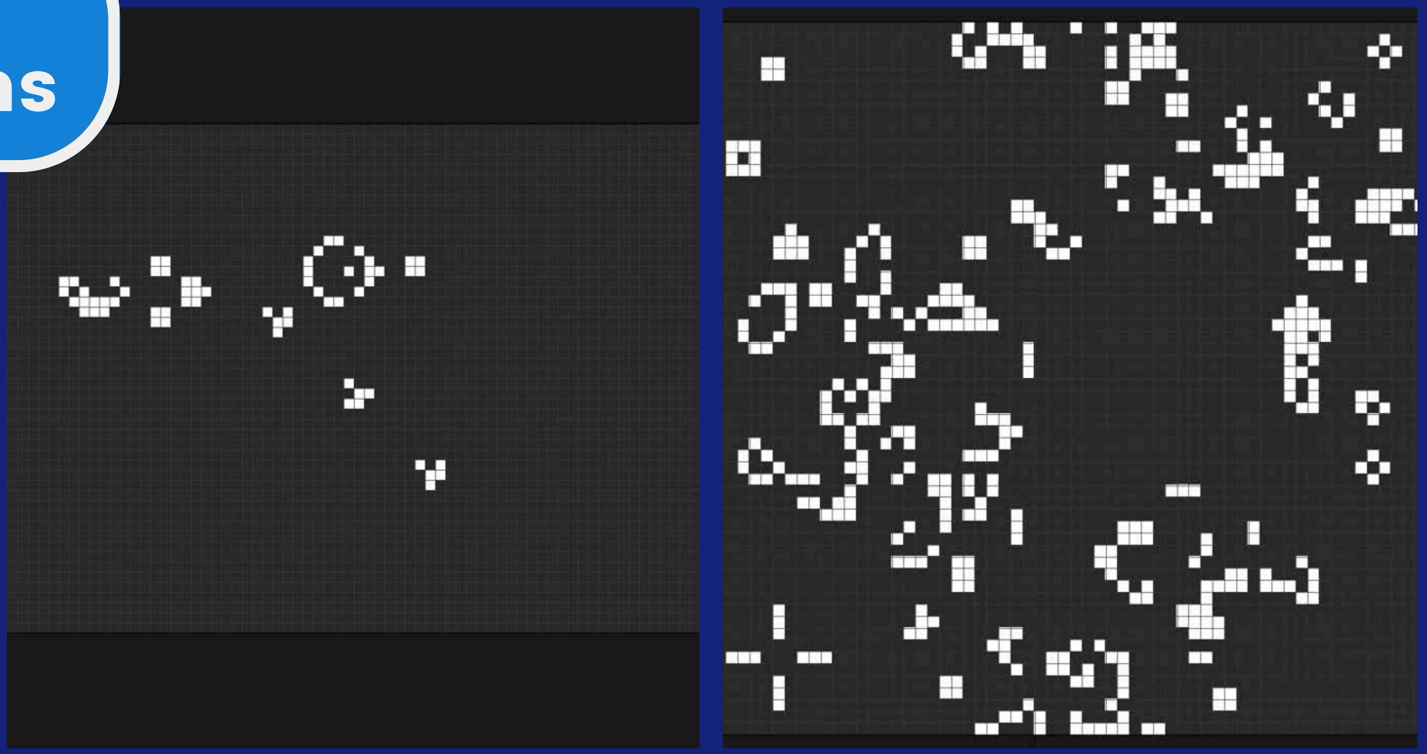
Setting:

- 2D grid of cells: each cell is alive or dead
- Neighborhood = 8 surrounding cells
- At each time the system updates.

Simple rules → complex patterns

Update rules:

- Alive stays alive if it has 2 or 3 live neighbors; otherwise it dies
- Dead becomes alive if it has exactly 3 live neighbors; otherwise stays dead



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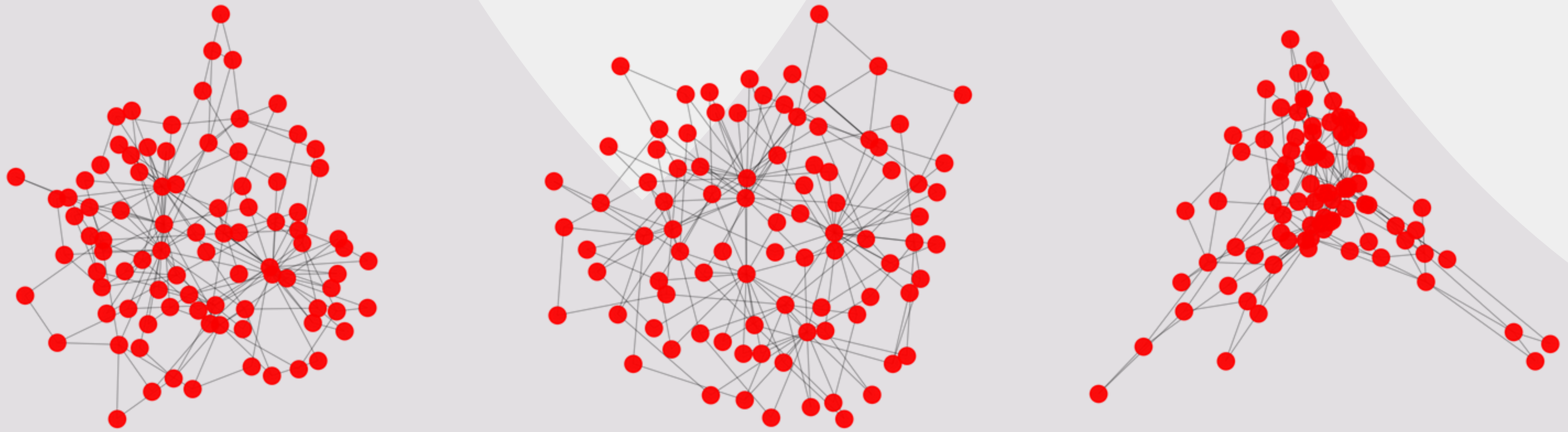
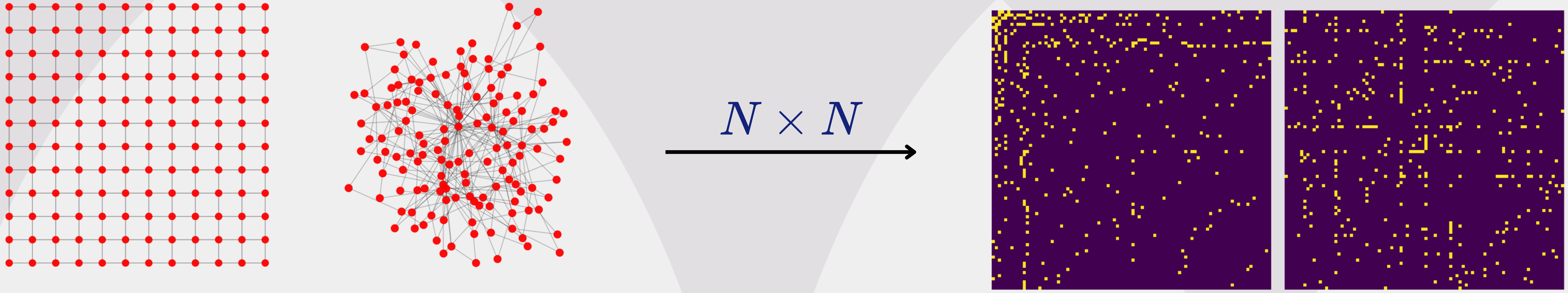
What is a network?

The general representation of Complex Systems' Structure

- no unique representation
- no clear notion of "geometry"





What is a network?

The intrinsic object is the abstract graph, encoded mathematically. $\dashrightarrow$ $G = (V, E)$

Adjacency Matrix $\dashrightarrow$

$$A_{ij} = \begin{cases} 1 & \text{if nodes } i \text{ and } j \text{ are connected} \\ 0 & \text{otherwise} \end{cases}$$

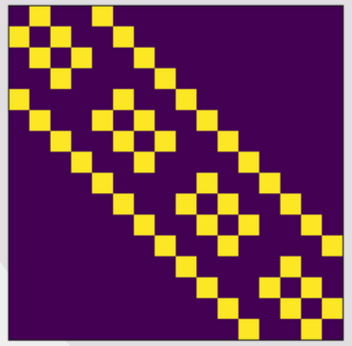
Edge list $\dashrightarrow$

$$E \; = \; \{\, (i, j) \in V \times V \; : \; A_{ij} = 1 \,\}$$

The case of a 4x4 square lattice

It's linear algebra after all

$$T_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Longrightarrow A = \begin{pmatrix} T_4 & I_4 & 0 & 0 \\ I_4 & T_4 & I_4 & 0 \\ 0 & I_4 & T_4 & I_4 \\ 0 & 0 & I_4 & T_4 \end{pmatrix}$$



Spectral Theory

Matrix = linear transformation. Studying matrix “effect” passes inevitably through the concept of spectrum.

- The spectrum of a matrix is given by the eigenvalues of the matrix
- Egenvalues corresponds to eigenvectors (or eigenmodes), “Special directions”
- Solutions of the *secular equation*

$$\sigma(A) = \{\lambda_i\} = \text{spectrum}$$

$$Ax = \lambda x$$

$$"A_{ij}x_j = \lambda_i x_i"$$

$$p_A(\lambda) = \det(A - \lambda I)$$

$$p_A(\lambda) = 0$$

**Caveat: all the results are for non-directed graphs, symmetry is required for ensuring the “diagonalizability” of matrices

Spectral Theory

- In the eigenvectors basis, matrix becomes diagonal,

$$Q = \begin{pmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_N \\ | & | & & | \end{pmatrix} \implies A = Q A_{\mathscr{B}} Q^\top$$

$$A_{\mathscr{B}} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{pmatrix} = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N)$$

- This allows for computing "functions" of the matrix, on the diagonal matrix ("independent numbers")

$$f(A) = Q\,\mathrm{diag}(f(\lambda_1), f(\lambda_2), \cdots, f(\lambda_N))Q^\top$$

E.g. say you want to compute the number of walks of length k between node i and node j. An easy counter is

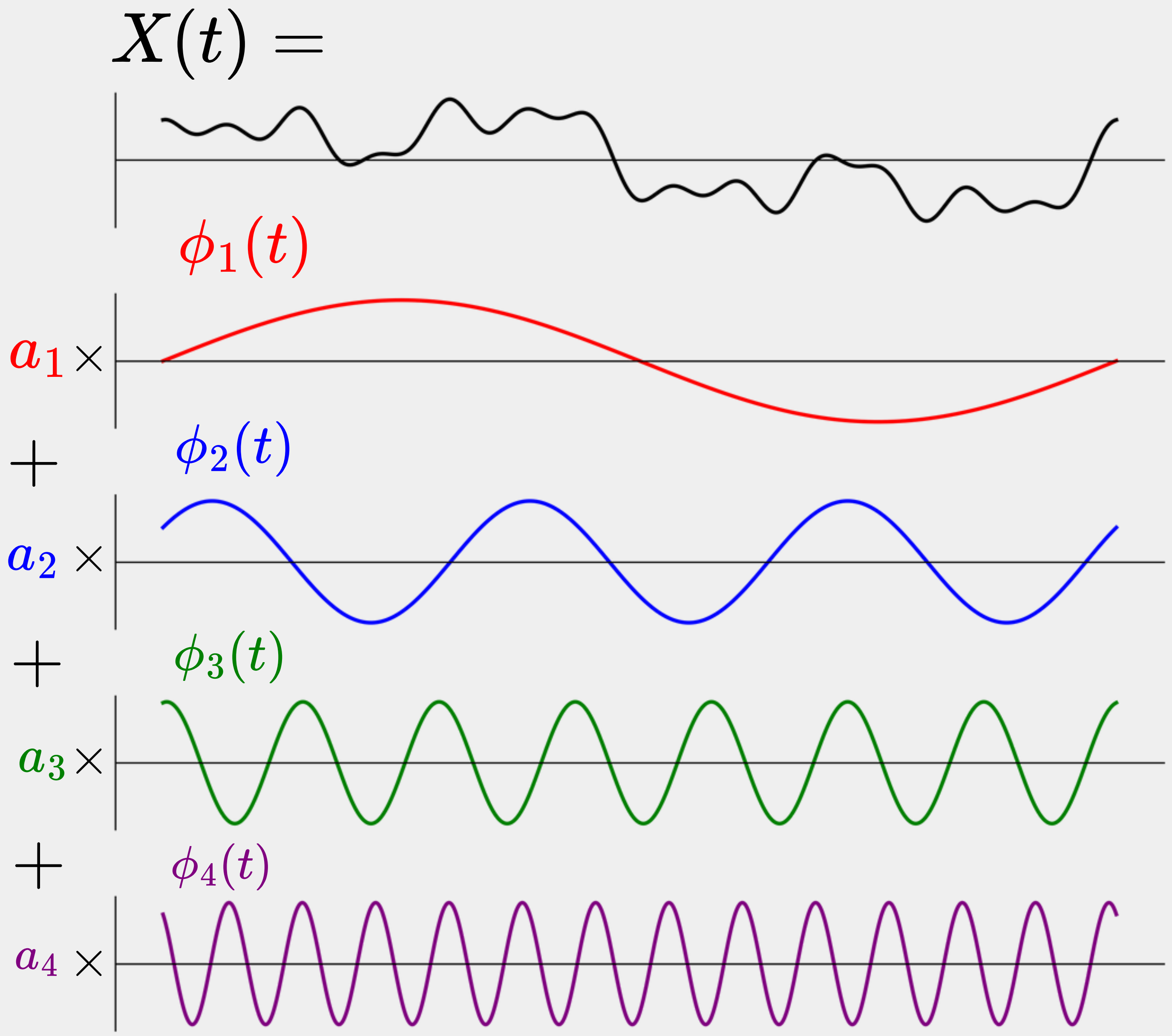
$$(A^k)_{ij}$$

To compute the k-th power of A we can leverage the power of spectral theory, indeed

$$A^k_{\mathscr{B}} = \mathrm{diag}(\lambda_1^k, \lambda_2^k, \cdots, \lambda_N^k)$$
$$A^k = Q A^k_{\mathscr{B}} Q^\top$$

E.g. number of triangles

$$\#\triangle = \frac{1}{6}\mathrm{Tr}(A^3) = \frac{1}{6}\sum_{r=1}^{N} \lambda_r^3.$$



Physical Modes

A complicated shape can be written as a sum of simple "modes" (eigenmodes).

- Given a signal we can decompose it into fundamental modes

$$X(t) = \sum_f a_f \times \phi_f(t)$$

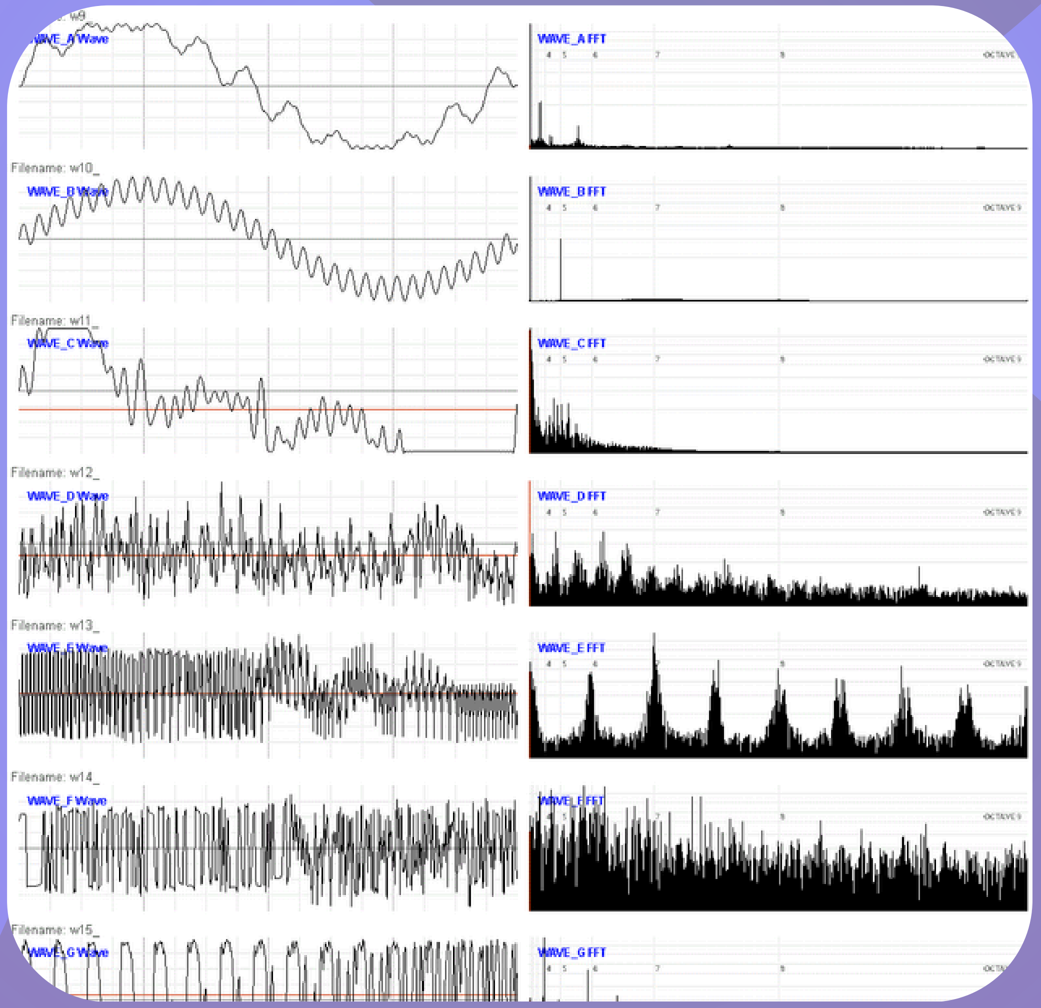
- Coefficients inform about intensity of components
- The "modes" (eigenmodes) here are "fundamental" comp. (e.g. sinusoids)
- Slow modes/fast modes → scale

writing spectrum = decomposition into simple modes (inverse space)

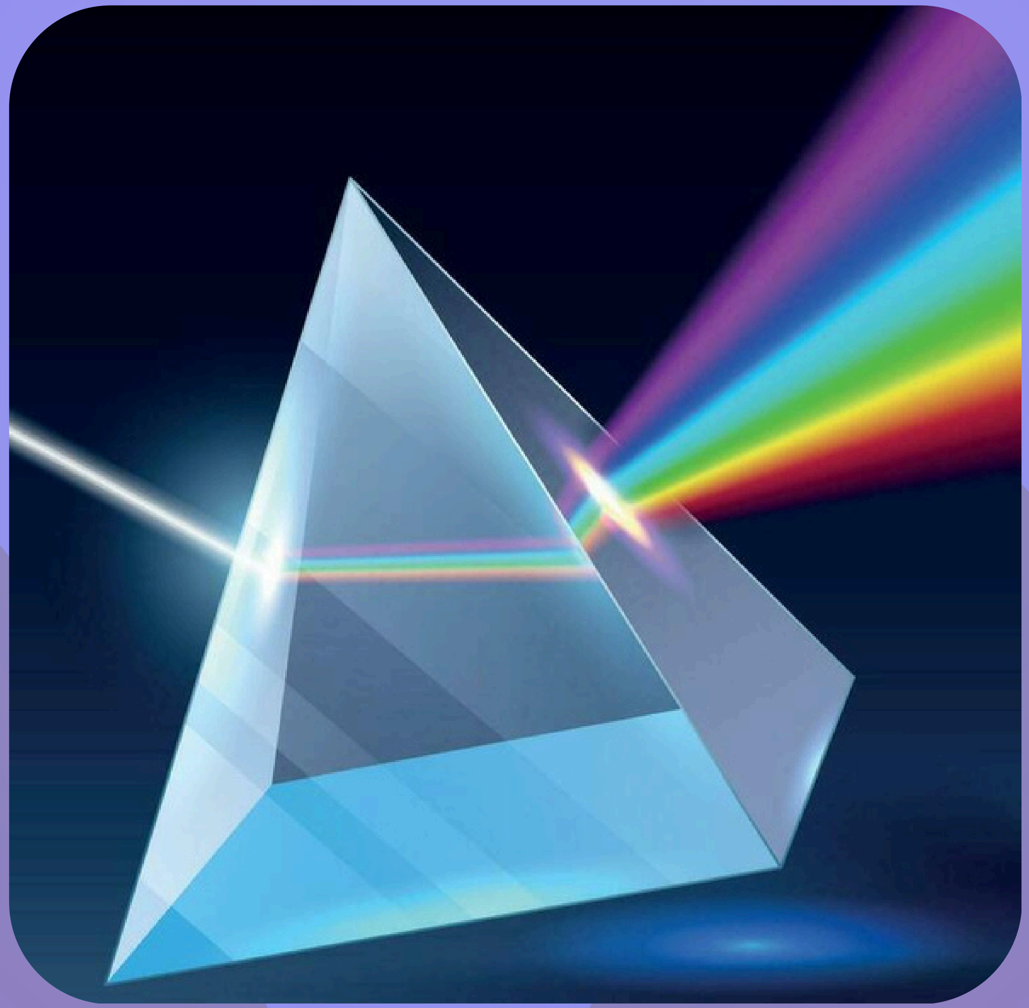
Physical Analogies

Physics can give a rich picture about “modes”

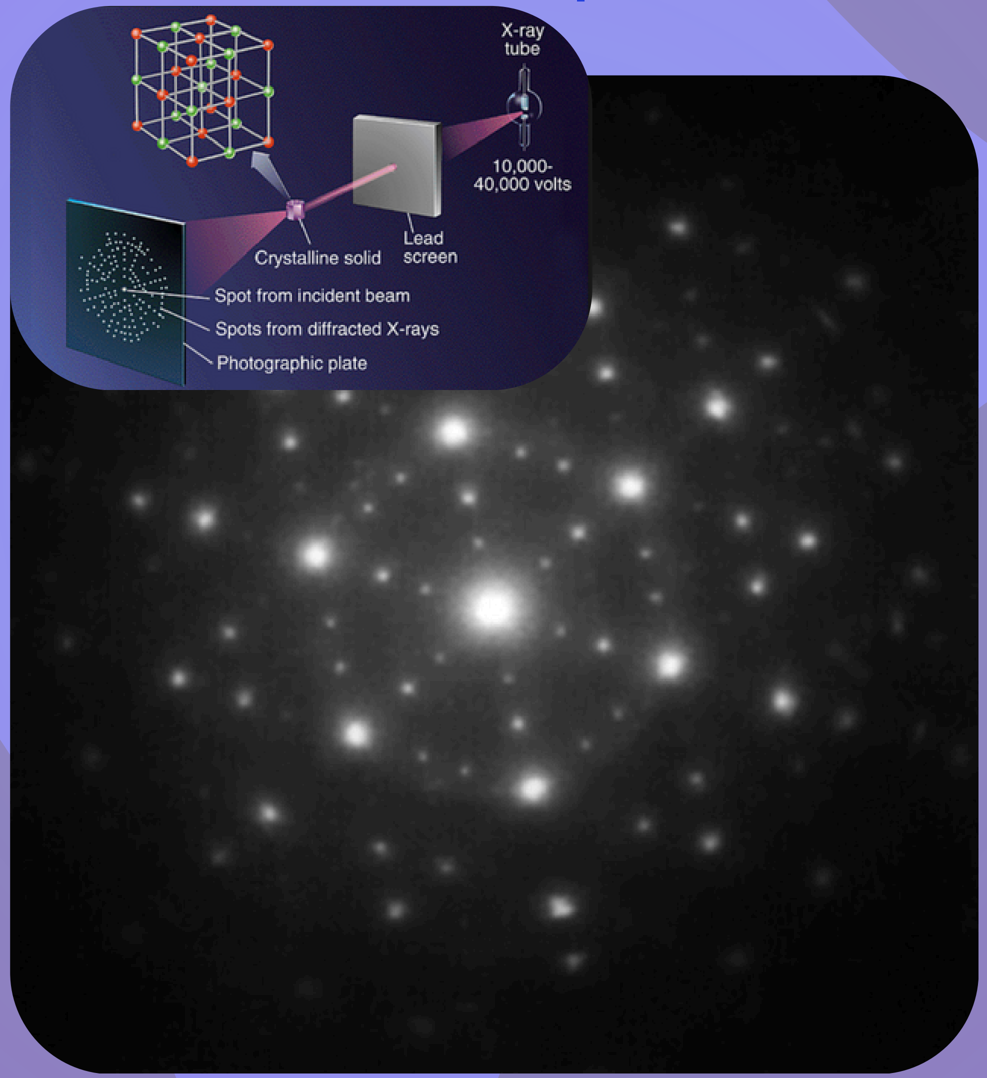
Sound → harmonics



White light → colors



Inverse space



Inverse space ≠ Upside Down!

Networks: Modes of what?

- Adjadcency matrix: *who is connected to whom*
- Laplacian turns connections into a **difference operator** on node states

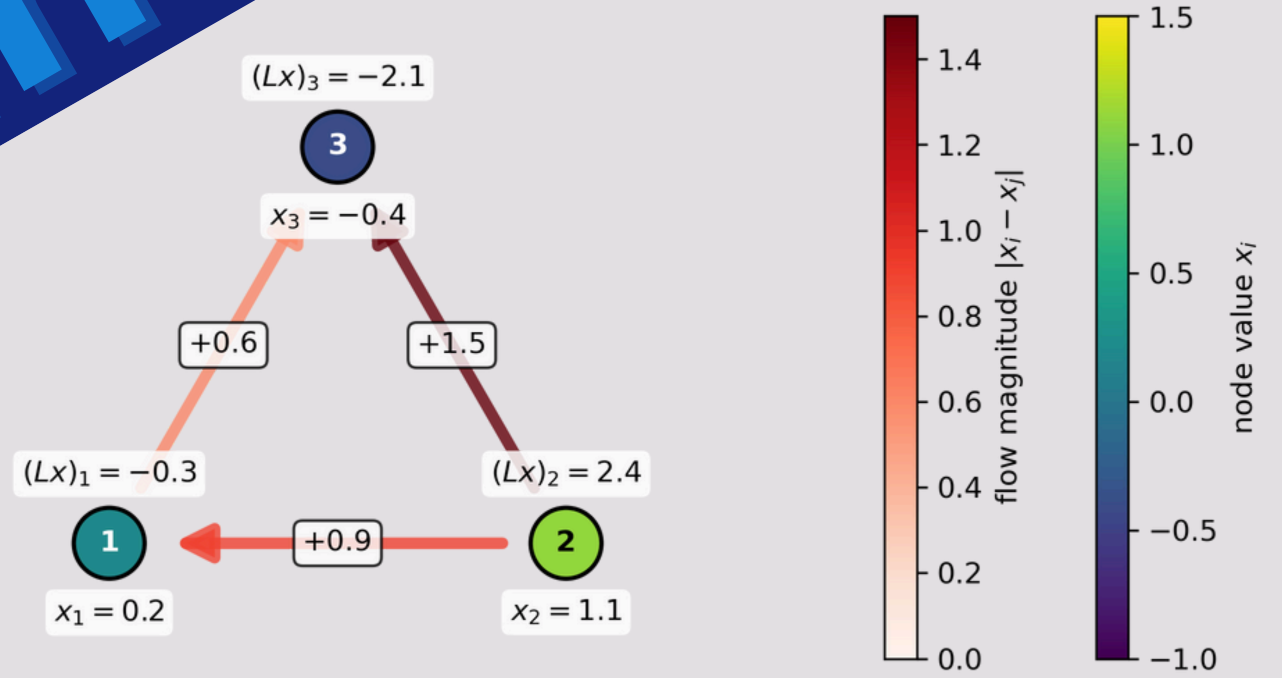
The graph Laplacian

$$L = D - A$$

$$D_{ii} = k_i = \sum_j A_{ij}$$

Assign one value to each node x_i (e.g. opinion, temperature, infection level, neural activity, traffic density).

$$(Lx)_i = \sum_j A_{ij}\,(x_i - x_j)$$



- Large (Lx)_i means x_i disagrees with its neighbors (sharp jump across edges)
- Laplacian = “how different are neighbors?”

The graph Laplacian

$$L = D - A$$

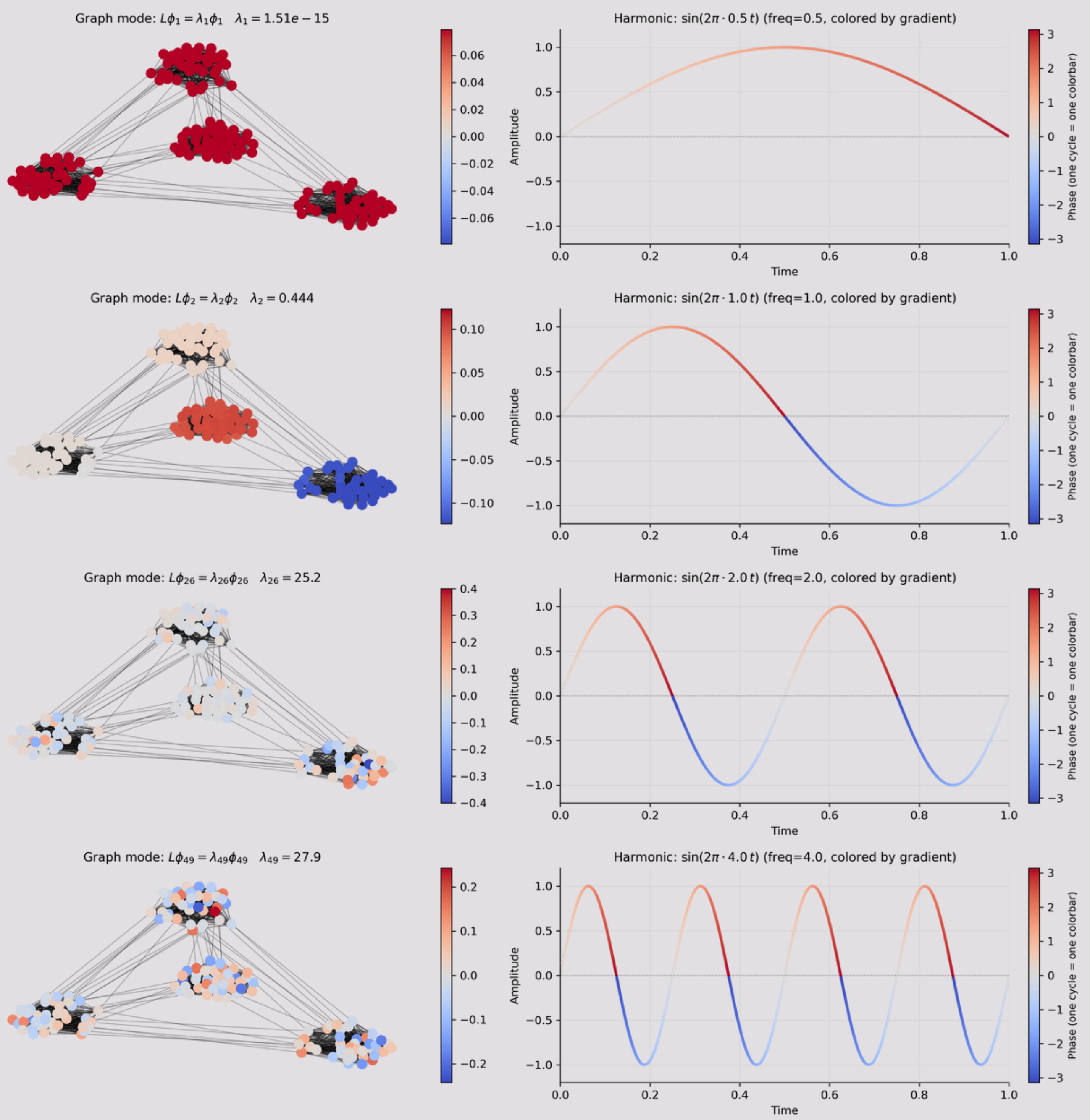
The Laplacian encodes information about "fundamental" graph connectivity patterns...

Souds familliar?

$$L\phi_k = \lambda_k \phi_k$$

ϕ_k : a pattern on nodes ("network harmonics")

Its spectrum is a preferential space for "patterns of information communication".



Diffusion

Laplacian is the operator describing the diffusion process

... But what is diffusion?

from thermodynamics (heat) to social system (opinions, infections...)
the process for which

flow goes from high to low, proportional to the difference

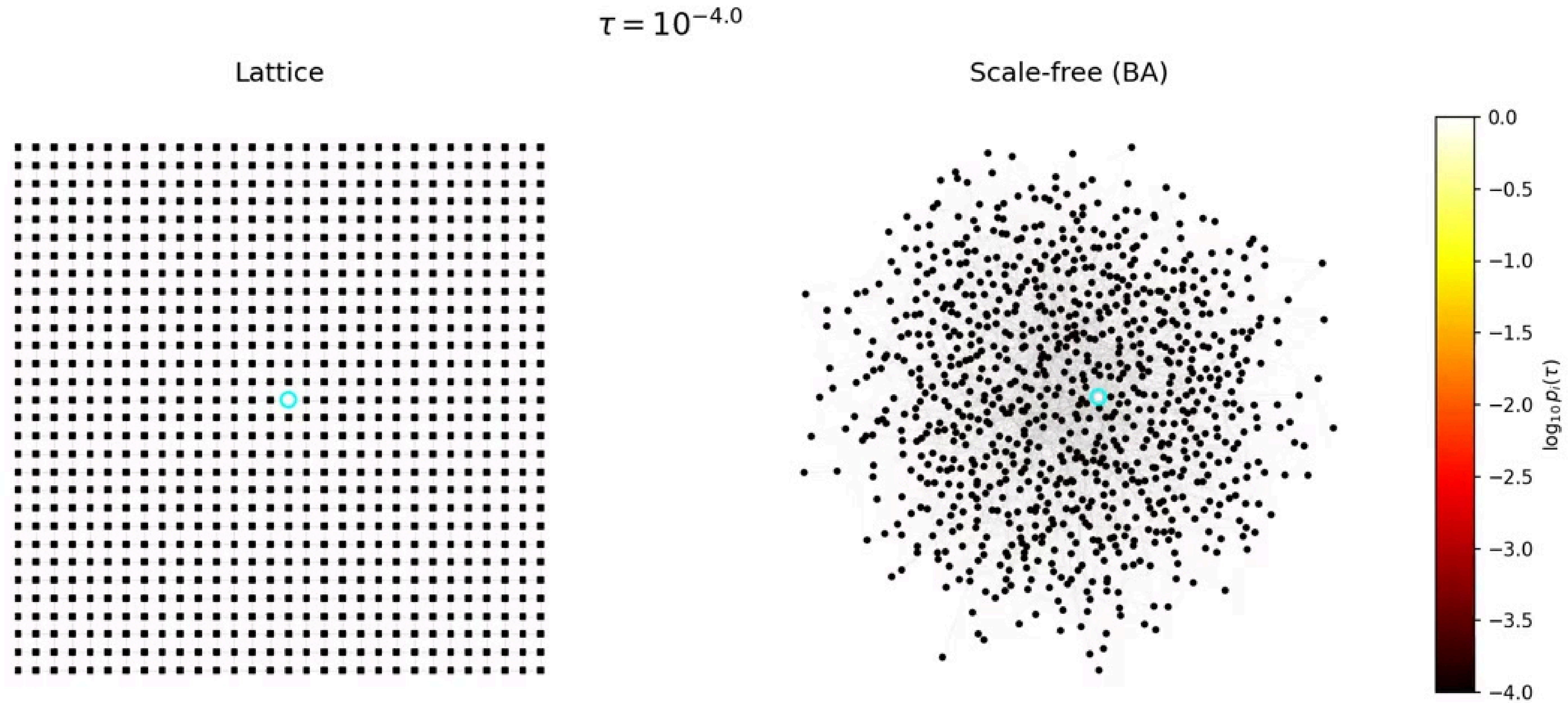
$$\frac{dx_i}{dt} = \sum_j A_{ij}\,(x_j - x_i) = -\sum_j L_{ij} x_j$$

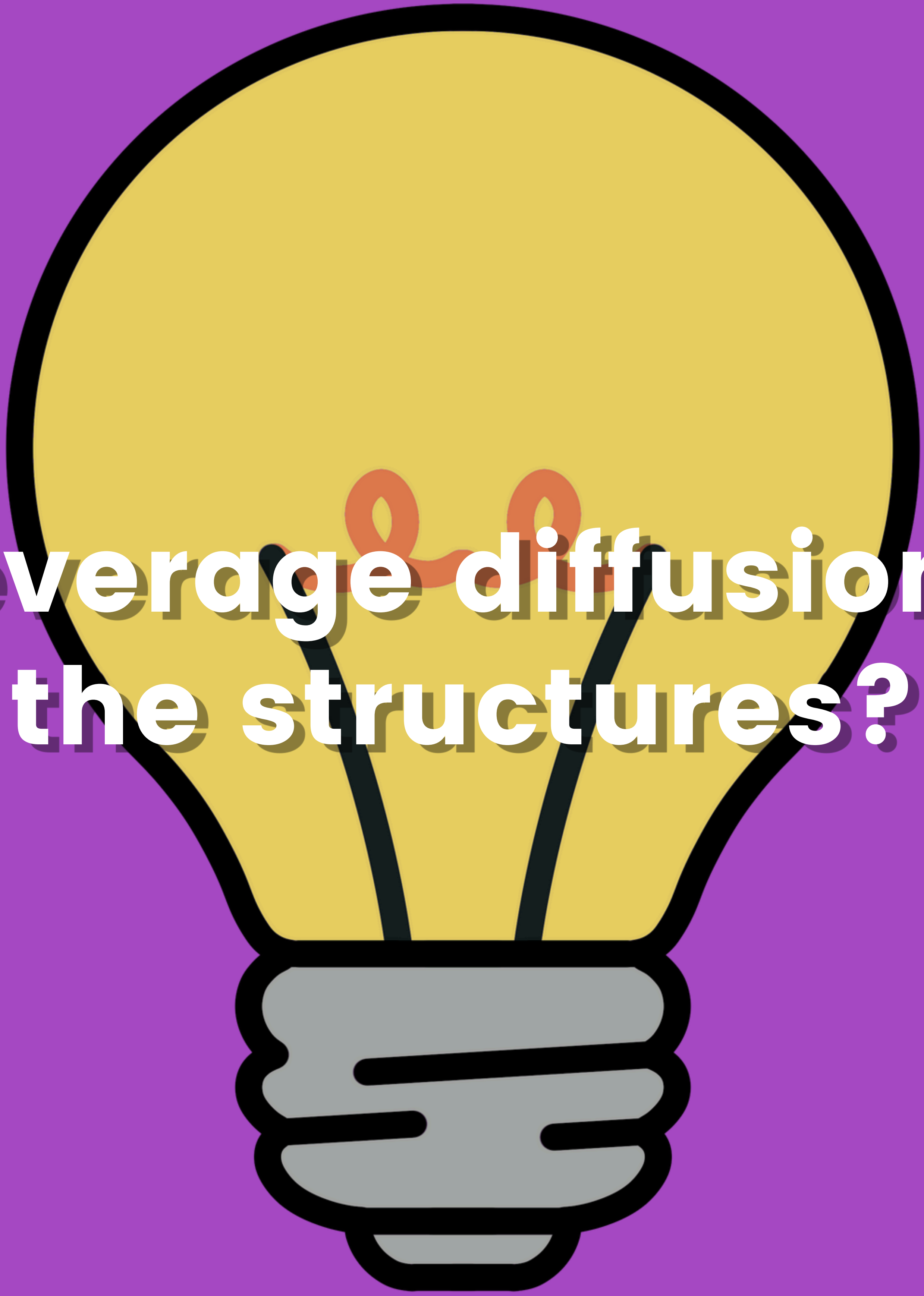
$$\dot{x} = -Lx$$

$$x(t) = e^{-\tau L}$$

Quiz time

On which structure the process will diffuse faster?





Can we leverage diffusion to probe the structures?

No.

The diffusion ensemble

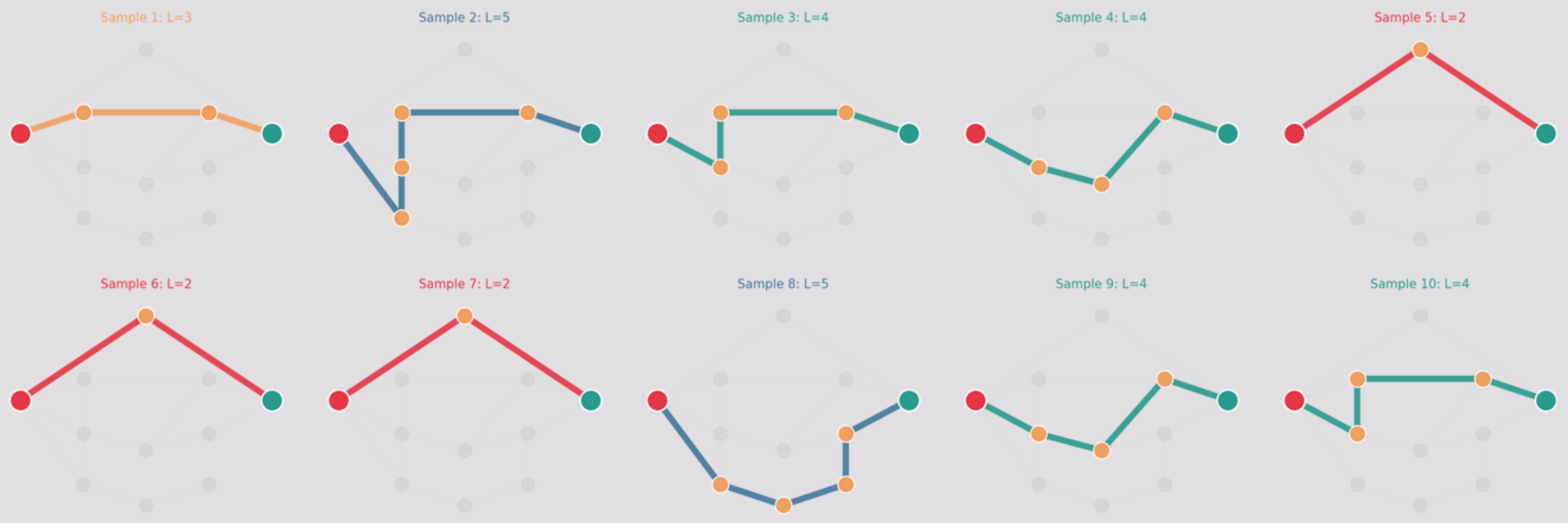
$$K(\tau) = e^{-\tau L} = Q \begin{pmatrix} e^{-\tau\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{-\tau\lambda_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{-\tau\lambda_N} \end{pmatrix} Q^\top$$

$$\rho(\tau) = \frac{K(\tau)}{\mathrm{Tr}(K(\tau))} = \frac{e^{-\tau L}}{\sum_i e^{-\tau\lambda_i}}$$

$$\rho_{ij}(\tau) = \mathrm{Prob}(\mathrm{i} \rightarrow \mathrm{j}) \text{ in time } \tau$$

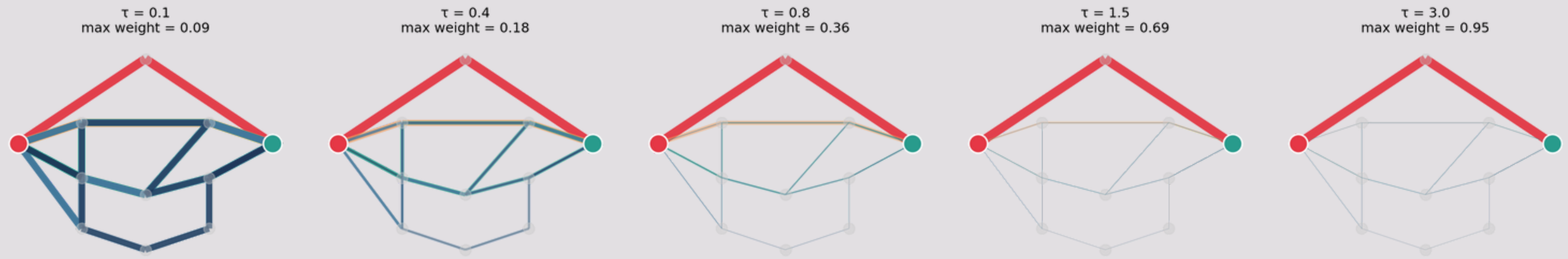
- The ensembles → probability distribution of paths of communication (transfer of information)
- Weights are exponential in eigenvectors
- Modes are patterns that “minimize the differences” between nodes at scale.

Ensemble samples weighted by $\rho_{ij}(\tau)$ — shorter paths dominate



Samples: L=2: 3, L=3: 1, L=4: 4, L=5: 2

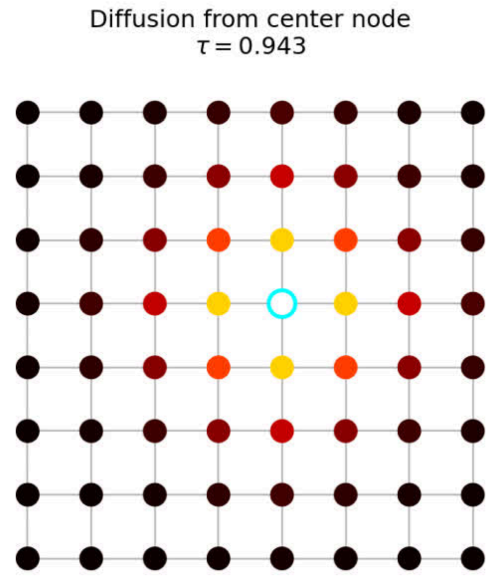
Path weight distribution evolves with τ: short paths dominate at large τ

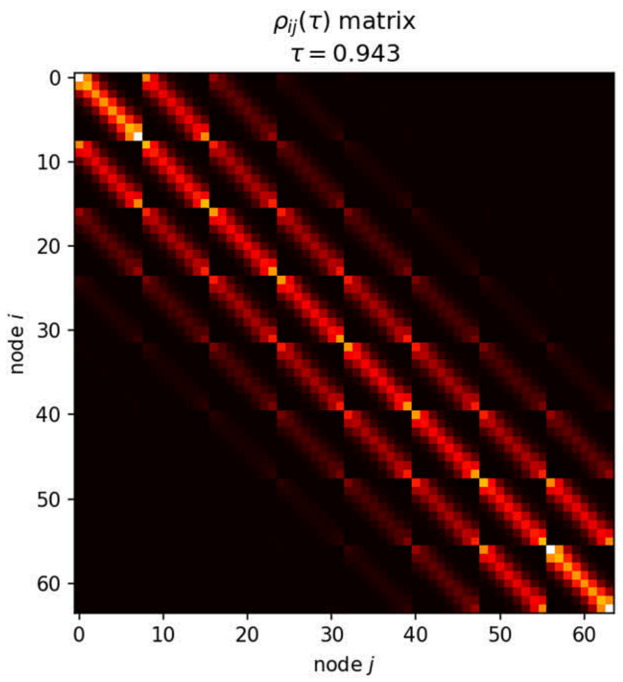


At longer times the ensemble suppresses progressively larger lambda, revealing the large scale behavior of the system (**slow modes**).

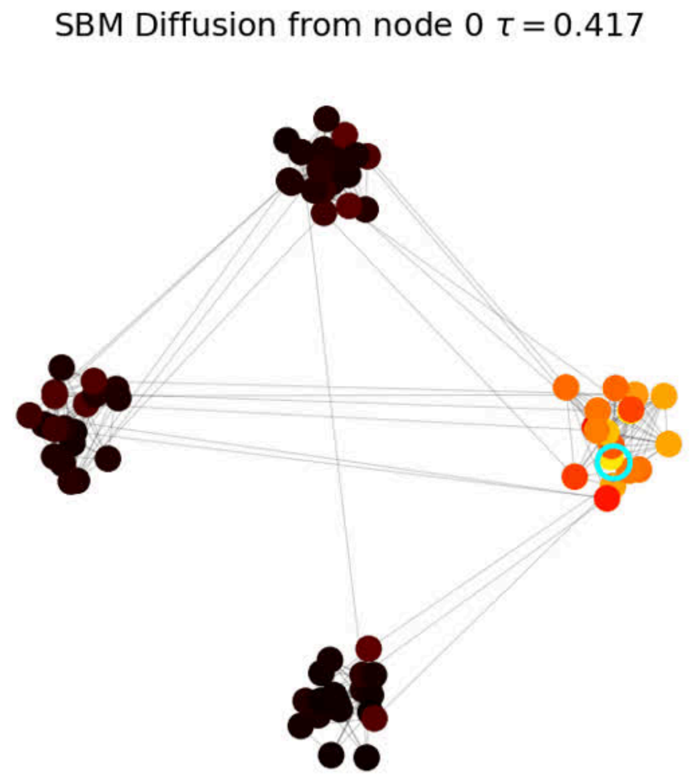
- Small lambda → large scales of the system, slow modes
- Large lambda → smaller scales, fast modes

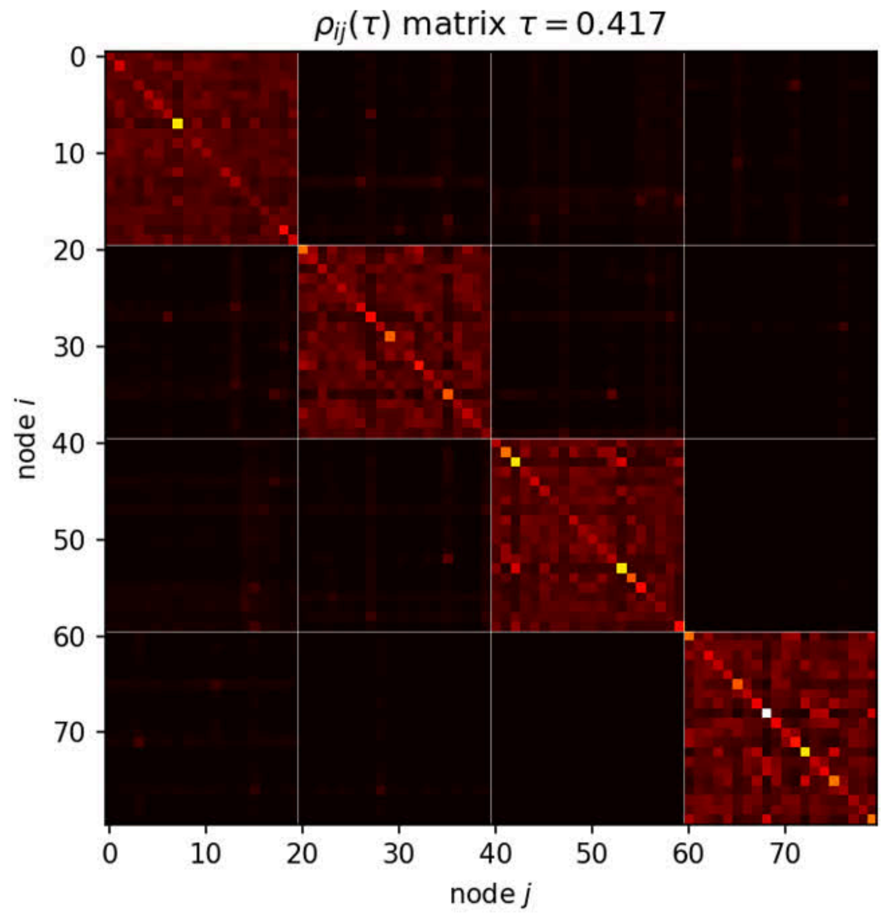
A spatio-temporal scanner

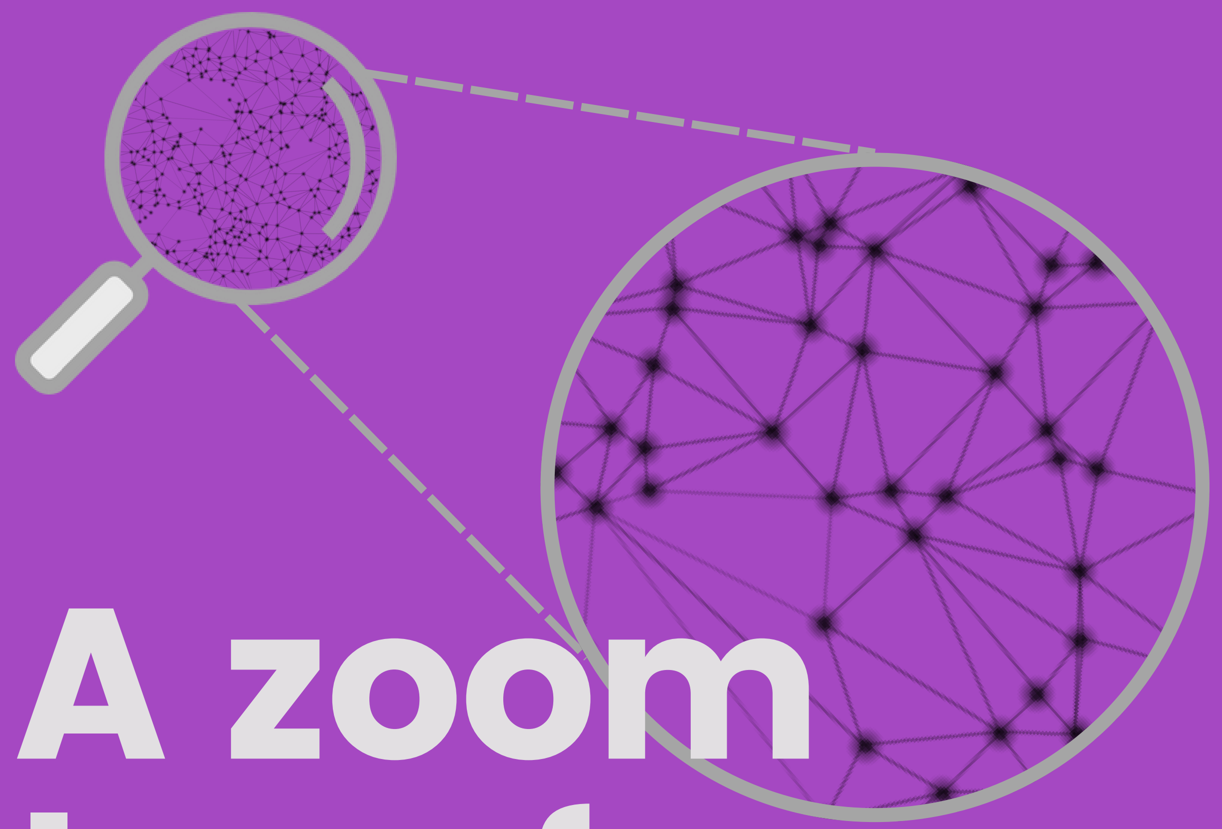




- decomposing in the system's intrinsic modes reveal scales and structures of the network
- intrinsic timescales → spatial scales (distance between nodes)







A zoom lens for network structures

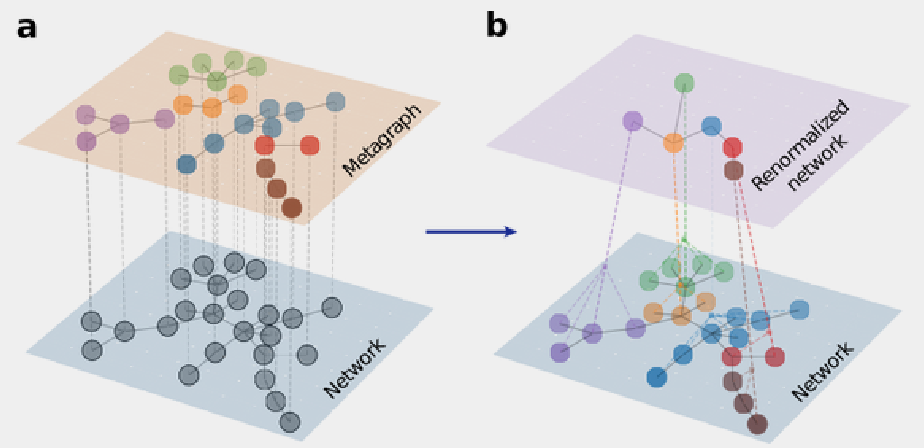
The Laplacian Renormalization Group offers a principled framework for compression and dimensionality reduction.

Real Space

- Pick a graph G
- Pick a timescale τ = τ*
- Build the ensemble ρ
- define distances

$$\mathcal{D}_{ij}(\tau^*) \propto \frac{1}{\rho_{ij}(\tau^*)}$$

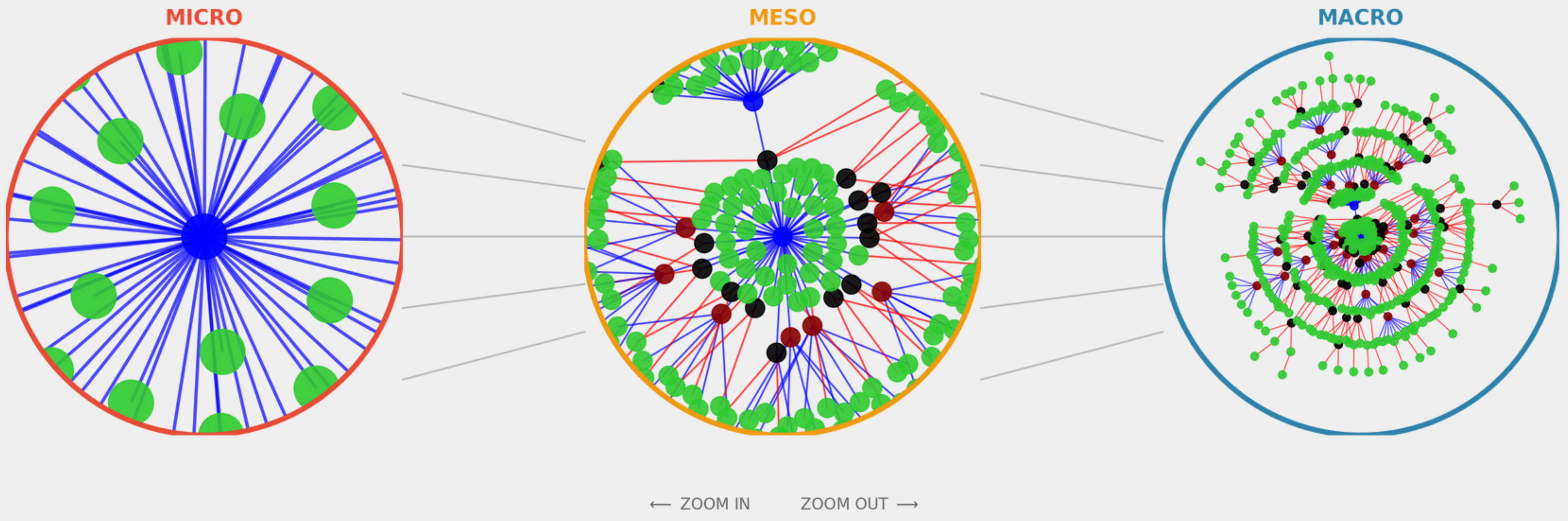
- Cluster/merge nodes i,j with smallest D(i,j;τ*) (largest communicability).



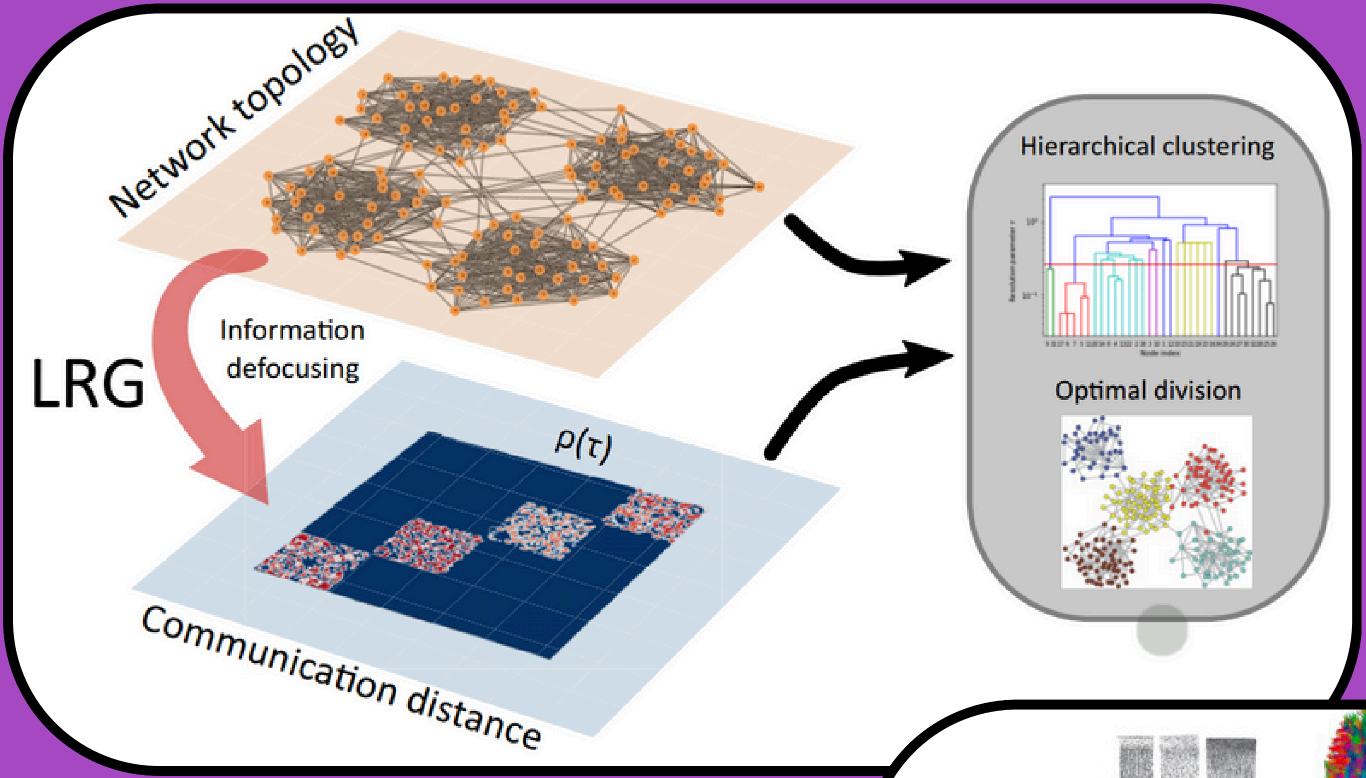
Inverse Space

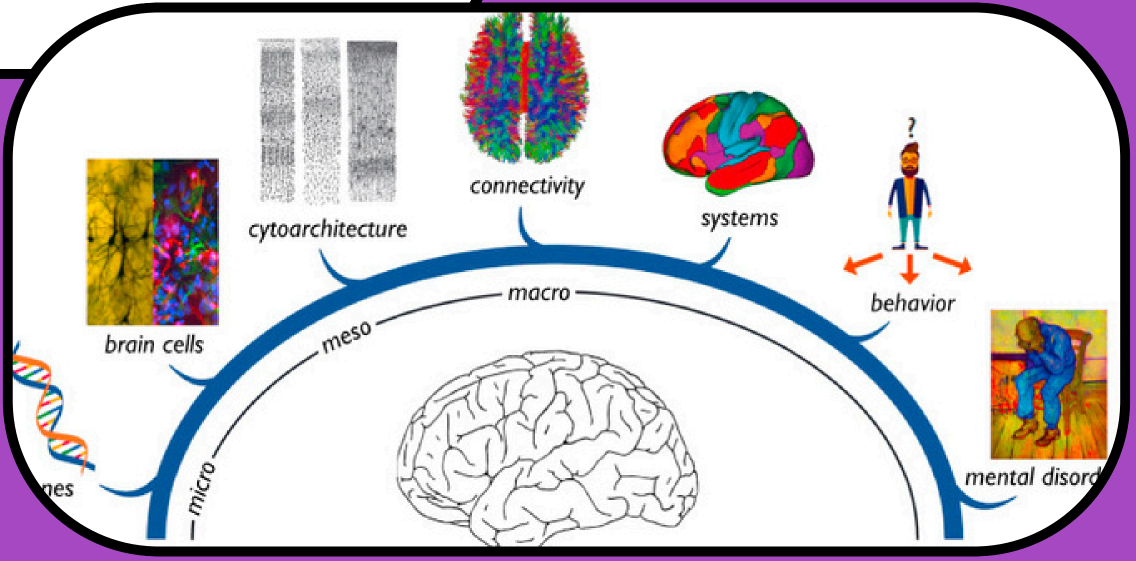
- Decompose Laplacian in modes
- Reduce+rescale L

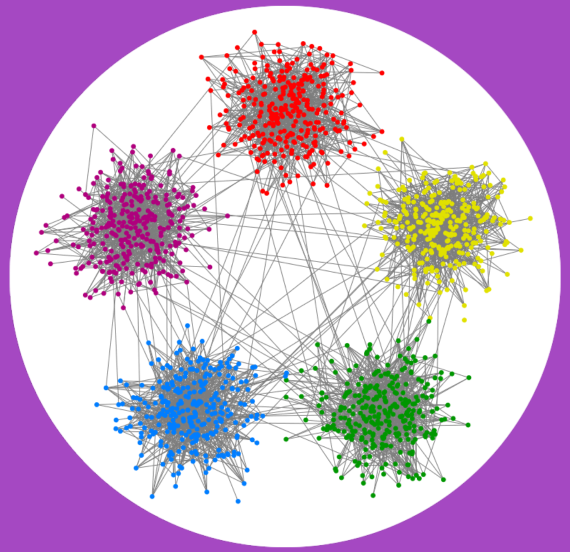
$$\lambda_* = \frac{1}{\tau_*}, \quad L' = \sum_{\lambda_k < \lambda_*} \lambda_k \phi_k \phi_k^\top, \quad L'' = \tau_* L'.$$

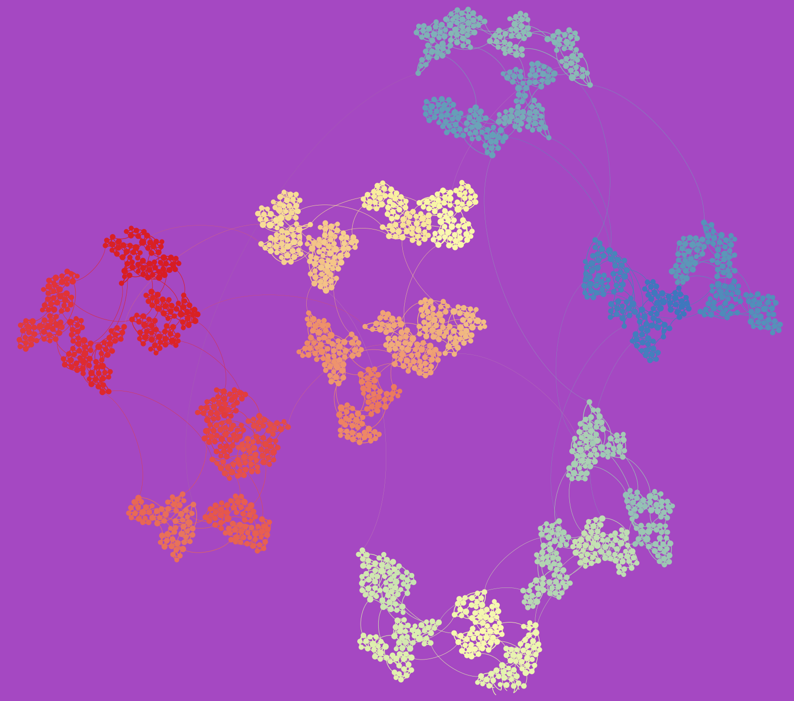


Communities can shift with scale, revealing higher-order structures, A multiscale answer for real complex network





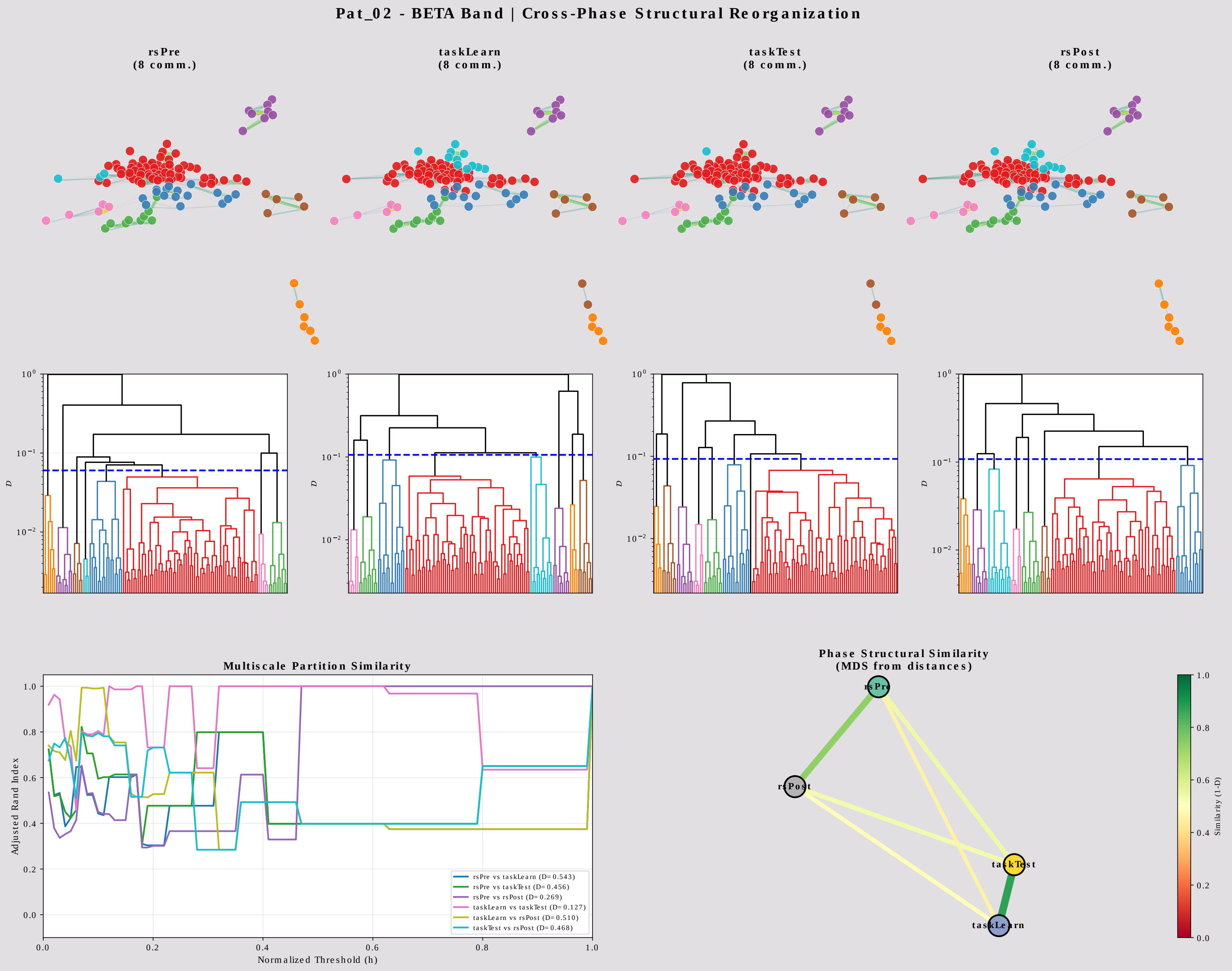




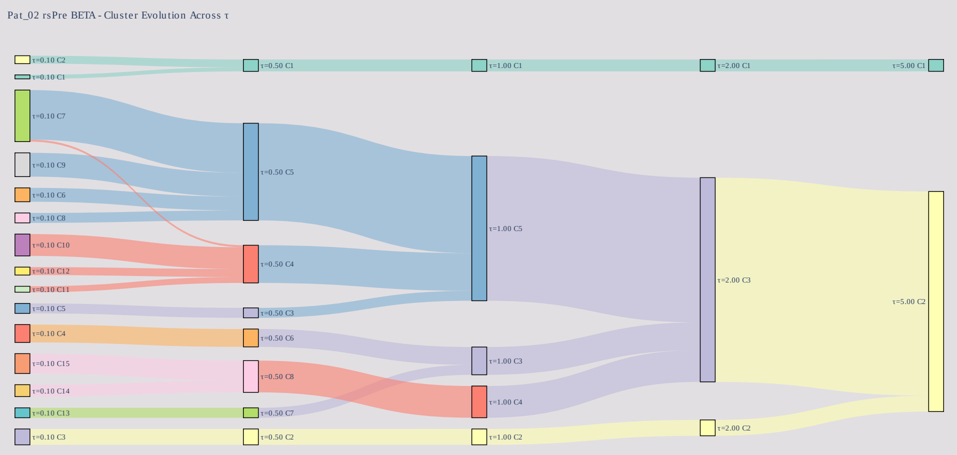
Mltiscale structure

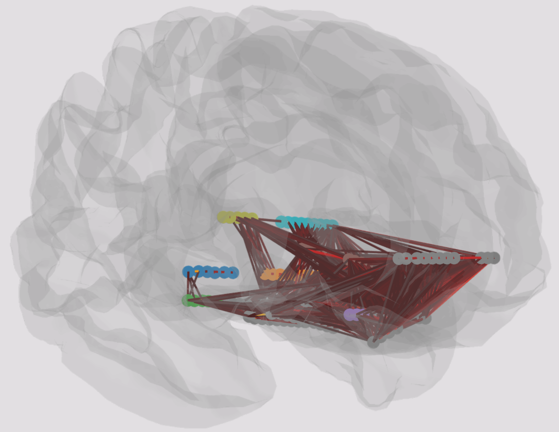
- Many complex systems (e.g. neural) present behaviors ldetermined by the presence of a hierarchy of structure.
- Uncover unprecedently seen phenomena and emergence explanation
- May inform on cross-scale interaction patterns.
- Multiscale schemes produce the entire hierarchical structure

Multiscale Structure in Brain Networks



- Functional connectivity networks from EEG signals to measure neural dynamics
- Experiment of cognitive task for epileptic patients
- Construct Multiscale partition and compare thorough the stages
- Pre/during/post task neurons reconfigure or not - based on the “frequency band”
- Metastable/bridge nodes connect scales





Easier to do it than to say it

- The whole process is basic linear algebra functions from numpy/scipy
- Easy to code
- Fast/efficient wrt to existing Multiscale methods
- A new quantitative bridge to frame presence and relevance of structures in networks into an interpretable inverse space

```python
import numpy as np
import networkx as nx
from scipy.sparse.linalg import expm
from scipy.spatial.distance import squareform
from scipy.cluster.hierarchy import linkage, dendrogram
import matplotlib.pyplot as plt


def laplacian_eigh(G: nx.Graph) -> tuple[np.ndarray, np.ndarray, list]:
    nodes = list(G.nodes())  # fixed node order
    L = nx.laplacian_matrix(G, nodelist=nodes).toarray()  # dense Laplacian
    eigv, eigV = np.linalg.eigh(L)  # full diagonalization
    return eigv, eigV, nodes


def rho_from_graph(G: nx.Graph, tau: float) -> tuple[np.ndarray, list]:
    nodes = list(G.nodes())  # fixed node order
    L = nx.laplacian_matrix(G, nodelist=nodes)  # sparse Laplacian
    K = expm(-tau * L)  # K = exp(-tau L)
    rho = (K / K.diagonal().sum()).toarray()  # rho = K / Tr(K), dense
    return rho, nodes


def plot_dendrogram_from_rho(
    rho: np.ndarray,
    labels: list | None = None,
) -> np.ndarray:
    D = 1.0 - np.asarray(rho, float)  # distance = 1 - affinity
    np.fill_diagonal(D, 0.0)  # zero diagonal
    y = squareform(D, checks=False)  # condensed distances
    Z = linkage(y, method="average")  # hierarchical clustering
    dendrogram(Z, labels=labels)  # plot
    plt.tight_layout()
    return Z
```

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2. Villegas, P., Gabrielli, A., Poggialini, A., & Gili, T. (2025). Multi-scale Laplacian community detection in heterogeneous networks. Phys. Rev. Res., 7, 013065. doi:10.1103/PhysRevResearch.7.013065
3. De Domenico, M., & Biamonte, J. (2016). Spectral Entropies as Information-Theoretic Tools for Complex Network Comparison. Phys. Rev. X, 6, 041062. doi:10.1103/PhysRevX.6.041062
4. Braunstein, S.L., Ghosh, S. & Severini, S. The Laplacian of a Graph as a Density Matrix: A Basic Combinatorial Approach to Separability of Mixed States. Ann. Comb. 10, 291–317 (2006). https://doi.org/10.1007/s00026-006-0289-3

Thank You!
Questions?